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A NEW 3-DIMENSIONAL SHRINKING CRITERION

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ABSTRACT. We introduce a new shrinking criterion for cell-like upper semicontinuous decompositions G of topological 3-manifolds, such that the embedding dimension (in the sense of Štan′ko) of the nondegeneracy set of G is at most one. As an immediate application, we prove a recognition theorem for 3-manifolds based on a new disjoint disks property.

1. INTRODUCTION

In 1979 M. Starbird developed two important shrinking criteria for 0-dimensional cell-like upper semicontinuous decompositions G of Euclidean 3-space E^3, called DDP I and DDP II (stands for "the disjoint disks property") [St]: G is said to have the DDP I if for all disjoint tame disks D_1, $D_2 \subset E^3$ so that $\partial D_1 \cup \partial D_2$ misses the nondegenerate elements of G, and for every open set $V \subset E^3$ which contains all the elements of G intersecting both D_1 and D_2, there are (1) a homeomorphism $g: E^3 \to E^3$ such that $g \mid E^3 - V = \mathrm{id}$ and (2) disks $D_1', D_2' \subset E^3$ obtained from $g(D_1)$ and $g(D_2)$, respectively, by replacement of subdisks so that each replacement subdisk used in getting from $g(D_i)$ to D_i', $i = 1, 2$, lies in V and so that no element of G intersects both D_1' and D_2'. If one can always assume that already $g(D_i) = D_i'$, $i = 1, 2$, then G is said to have the DDP II. Starbird proved in [St] that such decompositions are always shrinkable. If one replaces E^3 by an arbitrary topological 3-manifold M, then Starbird's result can be generalized as follows: the quotient space M/G of the decomposition G is a 3-manifold if and only if G has DDP I [Re2].

In the present paper we propose a new shrinking criterion called the *resolution disjoint disks property* (RDDP): a cell-like upper semicontinuous decomposition G of a topological 3-manifold M is said to have the RDDP if for every $\varepsilon > 0$, every $k \in \mathbf{N}$, and every collection of k pairwise disjoint, tame embeddings $f_i: B^2 \to M$, there exist maps $g_i: B^2 \to X = M/G$ satisfying (i) $\rho(g_i, \pi f_i) < \varepsilon$;

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and (ii) for every $i \neq j$, $g_i(B^2) \cap g_j(B^2) = \emptyset$. (One could define instead the RDDP as a property of the resolution $\pi: M \to X$ of the generalized 3-manifold X.) Our disjoint disks property applies to all those decompositions G whose nondegeneracy set N_G has embedding dimension (in the sense of M. A. Štan′ko [Št, Ed1]) at most one:

1.1. **Shrinking criterion.** *Let G be a cell-like, upper semicontinuous decomposition of a 3-manifold M such that $\operatorname{dem} N_G \leq 1$. Then G is shrinkable if and only if G has the RDDP.*

In the second part of this paper we apply this shrinking criterion to obtain another 3-dimensional recognition theorem. Recall that the recognition problem for topological n-manifolds asks for a list of simple geometric properties which a space X (usually assumed to be an ENR $\mathbf{Z}$-homology n-manifold) should possess in order to be a genuine n-manifold [Ca]. (For a review of the history of this problem see the survey [Re1].) We introduce a new general position property for generalized 3-manifolds, called the *light map separation property* (LMSP): a metric space (X, ρ) is said to have the LMSP if for every $\varepsilon > 0$, every $k \in \mathbf{N}$, and every map $f: B \to X$ of a collection of k standard 2-cells $B = \coprod_{i=1}^{k} B_i^2$ into X such that: (i) $N_f \subset \operatorname{Int} B$, where $N_f = \{y \in B \mid f^{-1}(f(y)) \neq y\}$; (ii) $\dim N_f \leq 0$; and (iii) $\dim Z_f \leq 0$, where $Z_f = \{x \in X \mid x \in f(B_i^2) \cap f(B_j^2)$ for some $i \neq j\}$; there exists a map $F: B \to X$ such that (1) $\rho(F, f) < \varepsilon$; (2) $F \mid \partial B = f \mid \partial B$; and (3) for every $i \neq j$, $F(B_i^2) \cap F(B_j^2) = \emptyset$. We first establish the (nontrivial) fact that every 3-manifold has the LMSP and then show that sometimes the LMSP can be applied to detect nonsingular spaces:

1.2. **Recognition theorem.** *A (metric) space X is a topological 3-manifold if and only if* (i) *X is the image under a proper cell-like map $f: M^3 \to X$, where M^3 is a 3-manifold and $\dim f(N_f) \leq 0$; and* (ii) *X has the LMSP.*

Edwards' celebrated shrinking theorem [Ed2] characterizes those cell-like maps $f: M^n \to X$ from an n-manifold ($n \geq 5$) to a finite-dimensional space that can be approximated by homeomorphisms, in terms of a disjoint disks property having the important quality of being measured solely in X. Striving to cast our results in the same vein, here we attempt to treat disjoint disks properties of X alone, unlike Starbird's, which pertain in a fundamental way to the domain and the explicit decomposition there. We are most successful with the LMSP, which certainly pertains just to X, while the RDDP is more of a hybrid, because the decomposition map is used to identify those singular disks that can be mapwise separated in X. Nevertheless, the RDDP has the useful feature, the significance of which is demonstrated in Edwards' argument, of being preserved under the operation of taking limits in the space of all (cell-like) maps $M \to X$. We should candidly acknowledge the negative side: that both the RDDP and the LMSP entail unpleasant complications by employing an arbitrary finite number of domain 2-cells, unlike the properties of [Ca, Ed2, St], which involve merely a pair of domain 2-cells.

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2. Preliminaries

We shall be working in the category of locally compact, metrizable spaces and continuous maps throughout the paper. Manifolds will be assumed to have no boundary unless specified. A space X is *cell-like* if there exists an n-manifold N and an embedding $f: X \to N$ such that $f(X)$ is *cellular* in N, i.e., $f(X) = \bigcap_{i=1}^{\infty} B_i^n$, where $\{B_i^n\}_{i\geq 1}$ is a properly nested decreasing sequence of n-cells in N. A map defined on an ANR X is *cell like* if its point-inverses are cell-like sets. A closed map is *proper* if its point inverses are compact. The *nondegeneracy set* of a map $f: X \to Y$ is the set $N_f = N(f) = \{x \in X \mid f^{-1}(f(x)) \neq x\}$. If for a subset $A \subset Y$, $N_f \cap f^{-1}(A) = \varnothing$, then we say that the map is *one-to-one* over A. A subset $Z \subset X$ *is locally simply coconnected* (1-LCC) if for every $x \in X$ and every neighborhood $U \subset X$ there is a neighborhood $V \subset U$ of x such that $\pi_1(V - Z) \to \pi_1(U - Z)$ is trivial.

Let G be a decomposition of a space X into compact subsets and let $\pi: X \to X/G$ be the corresponding quotient map, H_G the collection of all nondegenerate elements of G, and N_G their union (i.e., $N_G = N(\pi)$). A decomposition G is *upper semicontinuous* if π is a closed map. An upper semicontinuous decomposition G of a separable metrizable space X is *k-dimensional* if $\dim \pi(N_G) = k$, $k \in \mathbf{N}$.

A compactum $K \subset M^m$ in a PL m-manifold M has *embedding dimension* $\leq n$, written as $\operatorname{dem} K \leq n$, if for every closed subpolyhedron $L \subset M$ with $\dim L \leq m - n - 1$, there exist arbitrarily small ambient PL isotopies of M with support arbitrarily close to $K \cap L$, which move L off K [Ed1, Št].

A compact, contractible 3-manifold with boundary C is a *fake 3-cell* if C is not a topological 3-cell. A topological space X satisfies *Kneser finiteness* if no compact subset of X contains more than a finite number of pairwise disjoint fake 3-cells.

A space X is a *generalized n-manifold* ($n \in \mathbf{N}$) if (i) X is a euclidean neighborhood retract (ENR), i.e., for some integer m, X embeds in E^m as a retract of an open subset of E^m; and (ii) X is a homology n-manifold, i.e., for every $x \in X$, $H_*(X, X - x; \mathbf{Z}) \cong H_*(E^n, E^n - 0; \mathbf{Z})$. In dimension ≥ 3 X may fail to be locally euclidean at some (or perhaps all) points. We call such exceptions *singularities* of X and they form the *singular set* of X, $S(X) = \{x \in X \mid x$ has no neighborhood in X homeomorphic to $E^n\}$. Note that $S(X)$ is always closed and if $S(X) \neq X$ then $M(X) = X - S(X)$ is an open n-manifold. A *resolution* of an n-dimensional ANR X is a pair (M, f)

consisting of a topological n-manifold M and a proper, surjective cell-like map $f: M \to X$. Consequently, if X has a resolution then X is a generalized n-manifold [La1]. A resolution (M, f) of X can always be assumed to be *conservative*, i.e., the map f is one-to-one over $M(X)$. (See [BrLa] for $n = 3$, [Qu] for $n = 4$, and [Si] or [Ed2] for $n \geq 5$.)

Let X be a generalized 3-manifold with 0-dimensional singular set and let $p \in X$. Then p has arbitrarily small neighborhoods $N \subset X$ such that $\partial N \cap S(X) = \emptyset$ and ∂N is a closed orientable surface of some genus $n \geq 0$. If this n can always be $\leq m$, but not also $\leq m - 1$, then we say that the *genus of X at p* is equal to m, $g(X, p) = m$ [La2].

3. Proof of the shrinking criteron

3.1. **Lemma.** *Let M be a 3-manifold and G a cell-like, upper semicontinuous decomposition of M such that G has the RDDP and* $\operatorname{dem} N_G \leq 1$. *Then each $g \in G$ is cellular in M.*

Proof. Since $\operatorname{dem} N_G \leq 1$, each $g \in G$ has a neighborhood in M embeddable in E^3 [Ar, Lemma (5.3)]. Use the RDDP to prove that for every $x \in M/G$, $\{x\}$ is 1-LCC embedded in M/G. Therefore each $g = \pi^{-1}(x) \in G$ satisfies McMillan's Cellularity Criterion (see [Da, Corollary 18.4A]) and is thus cellular in M [Mc1].

3.2. **Lemma.** *Let G be a cell-like upper semicontinuous decomposition of a topological 3-manifold M with the RDDP and let $\pi: M \to M/G$ be the corresponding quotient map. Then for every $\varepsilon > 0$ and every finite collection $f_1, \dots, f_k: B^2 \to M$ of pairwise disjoint, tame embeddings satisfying $\pi f_i(\partial B^2) \cap \pi f_j(B^2) = \emptyset$ whenever $i \neq j$, there exist maps $g_1, \dots, g_k: B^2 \to M/G$ satisfying:*

(i) *for every i, $\rho(g_i, \pi f_i) < \varepsilon$;*
(ii) *for every i, $g_i \mid \partial B^2 = \pi f_i \mid \partial B^2$; and*
(iii) *for every $i \neq j$, $g_i(B^2) \cap g_j(B^2) = \emptyset$.*

Proof. This follows in a straightforward fashion by imposing motion controls and using either the fact that M/G is 1-LC or the combination of M/G being an ANR and a controlled version of the Borsuk Homotopy Extension Theorem [Bo].

We shall first prove the 0-dimensional special case of the Shrinking Criterion (1.1).

3.3. **Lemma.** *Let G be a 0-dimensional cell-like upper semicontinuous decomposition of 3-manifold M such that* $\operatorname{dem} N_G \leq 1$. *Then G is shrinkable if and only if G has the RDDP.*

Proof. The only if direction is obvious so we prove the other implication. Let $W \subset M$ be an open set containing N_G. It suffices to show that, for any two disjoint 2-cells B_1 and B_2 locally flatly embedded in M, there exists an embedding $h: B_1 \cup B_2 \to M$ such that h moves no point outside W, $h(W \cap (B_1 \cup B_2)) \subset W$, and $\pi h(B_1) \cap \pi h(B_2) = \varnothing$, (see Lemma 3.1 and [Št]).

Use the embedding dimension hypothesis to adjust the given 2-cells B_1 and B_2 slightly (with controls in M, not just in $X = M/G$) to achieve $(\partial B_1 \cup \partial B_2) \cap N_G = \varnothing$ and $\dim((B_1 \cup B_2) \cap N_G) \leq 0$.

Set $Z = \pi(B_1) \cap \pi(B_2)$. For every $x \in Z$ choose a neighborhood U_x with $U_x \subset \pi(W)$ and $U_x \cap \pi(\partial B_1 \cup \partial B_2) = \varnothing$. Since $Z \subset \pi(N_G)$, a 0-dimensional set, it is possible to extract a cover $\{V_i\}$ of Z refining $\{U_x\}$ and consisting of mutually exclusive open sets.

Find collections $D_1, \ldots, D_k$ (resp., $E_1, \ldots, E_n$) of pairwise disjoint 2-cells in B_1 (resp., B_2) whose interiors cover $\pi^{-1}(Z) \cap B_1$ (resp., $\pi^{-1}(Z) \cap B_2$), whose boundaries miss N_G, and whose images under π are contained in some element of the cover $\{V_i\}$. Note that for $i = 1, \ldots, k$, $\pi(\partial D_i)$ misses all the other singular disks $\pi(D_j)$. Hence, it is possible to extract a subdisk D_i^* of $\operatorname{Int} D_i$ whose boundary again misses N_G, whose interior contains $D_i \cap \pi^{-1}(Z)$, and which is large enough that $\pi(D_i - D_i^*)$ misses the other sets $\pi(D_j)$ (it necessarily misses $\bigcup \pi(E_j)$). Let E_i^* denote a subdisk of E_i with similar properties. Choose additional disks $D_{k+1}, \ldots, D_K$ and $E_{n+1}, \ldots, E_N$ in B_1 and B_2 disjoint from the others and subject to the same size controls, such that

$$\bigcup_{i=1}^{k} \pi^{-1}\pi(D_i^*) \subset \bigcup_{j=1}^{K} \operatorname{Int} D_j$$

and similarly for the disks E_i^* and E_j. Moreover, for $j \in \{k+1, \ldots, K\}$ require D_j to be contained in every $\pi^{-1}(V_i)$ such that some $D_s \subset \pi^{-1}(V_i)$, where $s \in \{1, \ldots, k\}$ and $\pi(D_s) \cap \pi(D_j) \neq \varnothing$, and require the same of the additional disks E_j in B_2. Define

$$P_1 = B_1 - \bigcup_{j=1}^{K} \operatorname{Int} D_j \quad \text{and} \quad P_2 = B_2 - \bigcup_{j=1}^{N} \operatorname{Int} E_j .$$

Choose a positive number δ less than each of the following:

$$\rho(\pi(P_1), \pi(D_i^*));$$
$$\rho(\pi(P_2), \pi(E_i^*));$$
$$\rho(\pi(D_i), X - V_s), \quad \text{where } V_s \text{ contains } \pi(D_i);$$
$$\rho(\pi(E_j), X - V_t), \quad \text{where } V_t \text{ contains } \pi(E_j);$$
$$\rho\left(\pi\left(B_1 - \bigcup \operatorname{Int} D_i^*\right), \pi(B_2)\right); \text{ and}$$
$$\rho\left(\pi(B_1), \pi\left(B_2 - \bigcup \operatorname{Int} E_i^*\right)\right).$$

Apply Lemma 3.2 to approximate $\pi \mid (\bigcup D_i^*) \cup (\bigcup E_i^*)$ by a map f agreeing with π on the various boundaries, with f δ-close to the restricted π, and with the images under f of these disks pairwise disjoint. Use the cell-likeness of π to approximately lift f to a map F of the same domain back into M, with F the identity on $\partial((\bigcup D_i^*) \cup (\bigcup E_i^*))$. There is no loss of generality in then assuming f was obtained with $\pi F = f$. Extend F to $F\colon B_1 \cup B_2 \to M$ via the inclusion elsewhere.

Now the idea is to invoke Dehn's Lemma for replacing F on each of the disks D_i (resp., E_i) by an embedding with the sort of properties allowing global reconstitution of B_1 (resp., B_2). The size controls above ensure that $\pi F(B_1) \cap \pi F(B_2) = \varnothing$ and that no two of the disks $f(D_i)$ (nor of the disks $f(E_i)$) intersect. According to Dehn's Lemma, there are tame disks $d_1, \dots, d_k$ and $e_1, \dots, e_n$ in $M - (P_1 \cup F(B_2)), M - (P_2 \cup F(B_1))$, respectively, with $\partial d_i = \partial D_i$ ($\partial e_i = \partial E_i$), with pairwise disjoint images under π, and with each image in the same element of $\{V_j\}$ as $\pi F(D_i)$ (or $\pi F(E_i)$). Now do disk trading, adjusting d_i and e_i, and remove all intersections of the resulting disks d_i' (resp., e_i') with $D_{k+1}, \dots, D_K$ (resp., $E_{n+1}, \dots, E_N$). Then

$$B' = \left(B_1 - \bigcup_{i=1}^{k} D_i\right) \cup \bigcup_{i=1}^{k} d_i' \quad \text{and} \quad B'' = \left(B_2 - \bigcup_{i=1}^{n} E_i\right) \cup \bigcup_{i=1}^{n} e_i'$$

are 2-cells in M whose images under π are disjoint.

The desired homeomorphism $h\colon B_1 \cup B_2 \to B' \cup B'' \subset M$ is one sending D_i onto d_i' (E_i onto e_i') and reducing to the identity elsewhere.

3.4. **Lemma.** *Let G be a cellular upper semicontinuous decomposition of a topological 3-manifold M such that G has the RDDP and* $\operatorname{dem} N_G \le 1$. *Let $A \subset M/G$ be a closed subset and denote by G_A the decomposition induced over A, i.e., $G_A = \{\pi^{-1}(a) \mid a \in A\} \cup \{\{x\} \mid x \in M - \pi^{-1}(A)\}$, where $\pi = M \to M/G$ is the quotient map. Then G_A is also upper semicontinuous, cellular, and has the RDDP.*

The proof is a routine lifting argument which exploits the induced cell-like map $p\colon M/G_A \to M/G$. One can work with disks $D_1, \dots, D_k$ in M for which $N_G \cap (\bigcup D_i)$ is 0-dimensional and obtain motion control in M/G_A by only lifting images of those 2-simplexes σ in some small mesh triangulation for which $\sigma \cap N_{G_A} \neq \varnothing$.

3.5. **Lemma.** *Let G be a cell-like decomposition of a 3-manifold M such that G has the RDDP, and let $\{h_i\colon M \to M \mid i = 1, 2, \dots\}$ be a sequence of homeomorphisms of M onto itself such that $\pi h_i\colon M \to M/G$ converges uniformly to a map $p\colon M \to M/G$. Then the decomposition $G_p = \{p^{-1}(x) \mid x \in M/G\}$ induced by p has the RDDP.*

Proof. Consider any collection of k pairwise disjoint, tame embeddings $f_i\colon B^2 \to M$. Given $\varepsilon > 0$, choose j sufficiently large that $\rho(\pi h_j, p) < \varepsilon/2$. Applying the RDDP to the embeddings $h_j f_i$ $(i = 1, \dots, k)$, one can find maps

$g_i: B^2 \to M/G$ having pairwise disjoint images and satisfying $\rho(g_i, \pi h_j f_i) < \varepsilon/2$. Clearly then $\rho(g_i, pf_i) < \varepsilon$.

Proof of (1.1). The only if direction is obvious so we prove the other implication. By [KoWa], $\dim Y = 3$ where $Y = M/G$. For classical reasons (see [Wa]), $\pi(N_G)$ is 1-dimensional. Hence, Y contains a 2-dimensional F_σ-set F such that $\dim(Y - F) = \dim(F \cap \pi(N_G)) = 0$. Express F as the union of compacta $A_i \subset Y$, $i \in \mathbf{N}$.

By construction and Lemmas 3.3 and 3.4, the decompositions G_i induced over A_i are shrinkable. As in [Ed2] (see [Da, Chapter 24]), $\pi: M \to Y$ can be approximated by a proper cell-like map $p: M \to Y$ such that p is one-to-one over F and $\operatorname{dem} N_p \leq 1$ (p arises as the limit of maps p_i, where p_i is one-to-one over A_i; given a sequence of triangulations T_j of M with mesh $T_j \to 0$ as $j \to \infty$, and $T_j^{(1)} \cap N_G = \varnothing$ (where $T_j^{(1)}$ is the 1-skeleton of T_j), one can choose p_i to be one-to-one over $p(T_j^{(1)})$, $1 \leq j \leq i$, and can impose controls so p is one-to-one over both $F = \bigcup_{i=1}^\infty A_i$ and $\bigcup_{i=1}^\infty p(T_i^{(1)})$). With Lemma 3.5 certifying that the 0-dimensional decomposition G_p has the RDDP, another application of Lemma 3.3 shows that p can be approximated by homeomorphisms. Thus, the same is true of π, or, equivalently, G is shrinkable [Da, Chapter 5].

3.6. **Corollary.** *Let X be a generalized 3-manifold with a resolution $\pi: M \to X$ such that $\operatorname{dem} N_\pi \leq 1$. Then X is a topological 3-manifold if and only if the decomposition $G = \{\pi^{-1}(x) \mid x \in X\}$ of M has the RDDP.*

4. Proof of the recognition theorem

4.1. **Lemma.** *Let G be a 0-dimensional cell-like upper semicontinuous decomposition of a 3-manifold M such that the quotient space M/G has the LMSP and each $g \in G$ has a neighborhood in M embeddable in E^3. Then G has the RDDP.*

Proof. By [DaRo], we may assume that $\operatorname{dem} N_G \leq 1$. Given any finite collection of pairwise disjoint, tame embeddings $f_i: B^2 \to M$ we can adjust them slightly, in M, so that $\dim(N_G \cap (\bigcup_{i=1}^k f_i(B^2))) \leq 0$. Then the map $f: B \to M/G$ given by $f = \coprod_{i=1}^k \pi f_i$, where $B = \coprod_{i=1}^k B_i^2$, defines the kind of map to which the LMSP applies, leading to a map $F: B \to M/G$ which shows G has the RDDP.

4.2. **Lemma.** *Every 3-manifold M has the LMSP.*

Proof. Consider a map $f: B \to M$ satisfying the hypotheses of LMSP and $\varepsilon > 0$. Using the hypothesis that $Z(f)$ is 0-dimensional, we successively determine compact 3-manifolds with boundary R, Q, and P satisfying:

(1) each component of R has diameter less than ε;
(2) $\operatorname{Int} R \supset Q \supset \operatorname{Int} Q \supset P \supset \operatorname{Int} P \supset Z(f)$;

(3) $\pi_1(P) \to \pi_1(Q)$ is trivial;
(4) B has a triangulation $\mathscr{T}$ with 1-skeleton

$$\Gamma \subset B - [N(f) \cup f^{-1}(P)];$$

(5) For each 2-simplex $\sigma \in \mathscr{T}$, $f(\sigma) \cap P \neq \varnothing$ implies $f(\sigma) \subset \operatorname{Int} Q$, and $f(\sigma) \cap Q \neq \varnothing$ implies $f(\sigma) \subset \operatorname{Int} R$.

The correct procedure is first to select R, Q, then to find $\mathscr{T}$ with $\Gamma \cap N(f) = \varnothing$, with $f(\sigma) \cap Z(f) \neq \varnothing$ implying $f(\sigma) \subset \operatorname{Int} Q$ and with $f(\sigma) \cap Q \neq \varnothing$ implying $f(\sigma) \subset \operatorname{Int} R$, and finally to identify P subject to (2)–(5).

We will verify the LMSP by adjusting f to a new map $F: B \to M$ such that F agrees with f on $\Gamma \cup [B - f^{-1}(R)]$, any two distinct points $F(x), f(x)$ belong to a component of R, and the images under F of the various disks B_i^2 comprising B are pairwise disjoint. In the course of these map adjustments we will also modify P, without changing Q or R, always maintaining (1)–(5) above. In particular, F will coincide with f on all 2-simplexes σ for which $f(\sigma) \cap Q = \varnothing$.

To get started, use the Simplicial Approximation Theorem and general position to make the map f PL on $f^{-1}(\operatorname{Int} Q) - \Gamma$, without changing f on Γ, in order to achieve the following:

(6) f is transverse to the 2-manifold ∂P.

Consider now the finite collection $\mathscr{C} = \{J \mid J$ is a simple closed curve from $f^{-1}(\partial P)\}$. Let $c(\partial P)$ be the complexity of ∂P defined by McMillan [Mc2],

$$c(\partial P) = \sum_{p \geq 0} (p+1)^2 g(p),$$

where $g(p)$ denotes the number of components of ∂P of genus p.

We show how to reduce $c(\partial P)$ to a minimum in a finite number of cut-and-paste operations, after which we obtain the map F by carefully trading singular disks in a modified $f(B)$ for others near ∂P.

For $L \in \mathscr{C}$ name the disk B_i^2 such that $L \subset B_i^2$, and let E_L denote the subdisk of B_i^2 bounded by L. Assume L is an innermost curve with respect to B_i^2. There are three cases to consider.

Case I. $f(L) \not\simeq *$ on ∂P and $f(E_L) \subset P$. Then apply the Loop Theorem to find an embedded disk $H \subset P - f(\Gamma)$ such that $H \cap \partial P = \partial H$ and $\partial H \cap f(\bigcup_{j \neq i} B_j^2) = \varnothing$. Thicken H to a 3-cell $C = H \times I$ in $\operatorname{Int} P$ for which $H = H \times \{1/2\}$ and $(\partial H) \times I \subset \partial P - f(\bigcup_{j \neq i} B_j^2)$. Redefine f on those 2-simplexes σ of $\bigcup_{j \neq i} B_j^2$ whose images meet C to eliminate such intersections, starting with innermost curves in the domain, so $f(\sigma) \subset Q - C$ and all new images lie in $P - C$. Make a compression of ∂P along H, forming a new P' in $P - \operatorname{Int} C$. This operation maintains conditions (1)–(5), and the redefinition of f ensures that the new singular set satisfies $Z(f) \subset \operatorname{Int} P'$.

Case II. $f(L) \not\simeq *$ on ∂P and $f(E_L) \subset \operatorname{Int} Q - \operatorname{Int} P$. Consequently, $f(E_L) \cap Z(f) = \emptyset$. Again use the Loop Theorem to obtain an embedding disk $H \subset \operatorname{Int} Q - f(\Gamma)$ such that $H \cap P = H \cap \partial P = \partial P$ and $\partial H \cap f(\bigcup_{j \neq i} B_j^2) = \emptyset$. Thicken H to a 3-cell $C = H \times I$, as before, with $C \subset \operatorname{Int} Q$ disjoint from $f(\Gamma) \cup f(\bigcup B_j^2)$ and with $C \cap \partial P = (\partial H) \times I$. Redefine f on those 2-simplexes σ of B_i^2 for which $f(\sigma) \cap P = \emptyset$ but $f(\sigma) \cap C \neq \emptyset$, in particular, on those where $f(\sigma) \not\subset Q$, so the new images lie in $\operatorname{Int} R - P \cup [\operatorname{Int} C \cup f(\bigcup_{j \neq i} B_j^2)]$. Make a compression of ∂P along H, forming a new P' in $P \cup \operatorname{Int} C$. This operation also maintains conditions (1)–(5), and here the redefinition of f is indispensible for obtaining (5).

Remark. According to [Mc2], $c(\partial P') < c(\partial P)$ in both Case I and Case II.

Case III. $f(L) \simeq *$ on ∂P. Consider the universal cover $p: \mathbf{H}^2 \to S$ where $S \subset \partial P$ is the component of ∂P containing $f(L)$. Then the loop $f(L)$ lifts into $\mathbf{H}^2$ as a collection of loops. Take one such lift $\gamma \subset p^{-1} f(L)$. It separates $\mathbf{H}^2$ into finitely many components $K_1 \dots, K_{r+1}$ where only the closure of K_{r+1} is noncompact.

Subcase III.a. For every $1 \le t \le r$, $f(\bigcup_{j \neq i} B_j^2) \cap p(K_t) = \emptyset$. Then we can cut $f(E_L)$ off from ∂P near $p(\bigcup_{t=1}^r K_t)$, eliminating the simple closed curve L from the collection $f^{-1}(\partial P)$, without introducing new singularities or new intersections with P. In light of the next subcase it is worth emphasizing that here L need not be innermost in B.

Subcase III.b. For some $1 \le t \le r$, $f^{-1}(f(\bigcup_{j \neq i} B_j^2) \cap p(K_t))$ contains a simple closed curve L'. Then L' has a special lift γ' to K_t, implying that L' falls under Case III and all but one of the components of $H^2 - \gamma'$ lie in K_t. Eventually we obtain a curve L' (not necessarily innermost with respect to B) for which Subcase III.a applies.

Finally, when $c(\partial P)$ is minimal, all curves $L \in \mathscr{C}$ must fall under Case III, which shows how they can all be eliminated via a new map $F: B \to M$ with $Z(F) \subset \operatorname{Int} P$ and $F(B) \cup P = \emptyset$. The subsequent images of the various disks B_i^2 are pairwise disjoint, as required.

Proof of (1.2). The forward implication follows immediately from Lemma 4.2. We concentrate on the reverse implication, where by [KoWa] X is 3-dimensional and thus by [La1] it is a generalized 3-manifold. Let $G = \{f^{-1}(x) | x \in X\}$ be the associated cell-like upper semicontinuous decomposition of M. Since $\dim \pi(N_G) = \dim f(N_f) \le 0$, G is 0-dimensional.

Let $C_0 = \bigcup\{g \in G \mid g$ has no neighborhood in M embeddable in $E^3\}$. Then by [ReLa2] the set $f(C_0)$ is locally finite in X. Let G_0 denote the (cell-like) decomposition of M consisting of the components of C_0 and the singletons from $M - C_0$. Consider $M_1 = M/G_0$ and the associated decomposition $G_1 = \pi_0(G) = \{\pi_0(g) \mid g \in G\}$ of M_1, where $\pi_0 \colon M \to M_1$ is the quotient map. Clearly M_1 is a generalized 3-manifold and $S(M_1) \subset \pi_0(C_0)$.

Assertion. *$X - f(C_0)$ is a 3-manifold.*

Proof. Every $g' \in G_1' = \{g \in G_1 \mid g \subset M_1 - \pi_0(C_0)\}$ has a neighborhood in $M_1' = M_1 - \pi_0(C_0)$ embeddable in E^3. Let $\pi_1 \colon M_1 \to X$ be the quotient map of the decomposition G_1, and set $\pi_1' = \pi_1 \mid M_1'$. So (M_1', π_1') is a resolution of $X' = X - f(C_0)$. Since X has the LMSP, so does X'. Hence Lemma 4.1 applies, implying G_1' has the RDDP, and by Lemma 3.3 G_1' is shrinkable. This confirms the assertion.

By [BrLa] we can assume that f is one-to-one over X'. Based on LMSP and the existence of f, it is a simple matter to verify that each $f(c) \in f(C_0)$ is 1-LCC embedded in X (see the proof of [ReLa1, Theorem 3.1]). By Theorem 4 of [BrLa], X is a 3-manifold.

By way of application we have another recognition theorem:

4.3. **Corollary.** *A space X is a 3-manifold if and only if it satisfies the following properties:*

(i) *each $x \in X$ is 1-LCC embedded in X;*
(ii) *X admits a resolution $\pi \colon M^3 \to X$ defined on a 3-manifold;*
(iii) *$S(X)$ is contained in a finite graph Γ (topologically) embedded in X;*
(iv) *X has the LMSP.*

Proof. In case X satisfies properties (i)–(iv), we can assume the resolution $\pi \colon M^3 \to X$ is one-to-one over $X - \Gamma$. Let

$$E = \{x \in X \mid \pi^{-1}(x) \text{ has no neighborhood that embeds in } E^3\}.$$

For the reasons set forth in the proof of (1.2), E is a discrete subset of X. Select a countable dense subset D of $\Gamma - E$. As in the proof of Lemma 3.1, each $\pi^{-1}(d)$, $d \in D$, is cellular in M; consequently, we can approximate π by a cell-like map $f \colon M^3 \to X$ such that f is one-to-one over $D \cup (X - \Gamma)$. This verifies that X satisfies the conditions of (1.2), which in turn shows X is a 3-manifold.

Virtually the identical argument yields the next result, an improvement to Corollary 4.3.

4.4. **Corollary.** *A space X is a 3-manifold if and only if it satisfies the following properties:*

(i) *X admits a resolution $\pi \colon M^3 \to X$ defined on a 3-manifold;*
(ii) *each $s \in S(X)$ has arbitrarily small neighborhoods whose frontiers B_s are such that $\dim[B_s \cap S(X)] \le 0$ and $B_s \cap S(X)$ is 1-LCC embedded in X;*
(iii) *X has the LMSP.*

5. Epilogue

We close by spelling out some unresolved issues. The first pertains to potential improvements to Shrinking Theorem (1.1).

5.1. **Conjecture.** *If $\pi: M^3 \to X$ is a resolution of X with the RDDP, then X is a 3-manifold.*

The fundamental difficulty occurs in examining decompositions induced over closed subsets.

5.2. **Conjecture.** *If G is a cell-like decomposition of a 3-manifold M such that G has the RDDP and if A is a subset of M/G, then the decomposition G_A induced over A has the RDDP.*

In our attempts to improve on the Recognition Theorem (1.2), we repeatedly encountered some form of the problem stated below.

5.3. **Conjecture.** *Every 3-manifold has the LMSP*, where LMSP* stands for the LMSP without any hypothesis on the set $Z(f)$.*

Only if Conjecture 5.3 is true does the next one make sense.

5.4. **Conjecture.** *A space X is a 3-manifold if X has the LMSP* and it admits a resolution $\pi: M^3 \to X$ defined on a 3-manifold.*

Finally, it seems that a stronger result than 5.3 might be valid. Compare with [An].

5.5. **Conjecture.** *Let $f: S^2 \to M$ be a map of a 2-sphere into a 3-manifold such that N_f is 0-dimensional. Then for each $\varepsilon > 0$ there exists an embedding $F_\varepsilon: S^2 \to M$ such that $\rho(F_\varepsilon, f) < \varepsilon$.*

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